

Joint source-channel coding for a quantum multiple access channel

Mark M. Wilde and Ivan Savov

School of Computer Science, McGill University, Montreal, Quebec H3A 2A7

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Abstract

Suppose that two senders each obtain one share of the output of a classical, bivariate, correlated information source. They would like to transmit the correlated source to a receiver using a quantum multiple access channel. In prior work, Cover, El Gamal, and Salehi provided a combined source-channel coding strategy for a classical multiple access channel which outperforms the simpler “separation” strategy where separate codebooks are used for the source coding and the channel coding tasks. In the present paper, we prove that a coding strategy similar to the Cover-El Gamal-Salehi strategy and a corresponding quantum simultaneous decoder allow for the reliable transmission of a source over a quantum multiple access channel, as long as a set of information inequalities involving the Holevo quantity hold.

Suppose that a correlated information source, embodied in many realizations of two random variables U and V distributed independently and identically as $p(u, v)$, is in the possession of two spatially separated senders, such that one sender has U and the other V . Suppose further that a multiple access channel, modeled as a conditional probability distribution $p(y|x_1, x_2)$, connects these two senders to a single receiver. When is it possible for the senders to transmit the output of the correlated source reliably over the multiple access channel? A *separation* strategy would have the senders first use a Slepian-Wolf compression code [26] to compress the source, followed by encoding the compressed bits using a multiple access channel code [14, 1]. The receiver would first decode the multiple access channel code and then decode the Slepian-Wolf compression code in order to recover the output of the information source. This strategy will work provided that the following information inequalities hold:

$$\begin{aligned} H(U|V) &\leq I(X_1; Y|X_2), \\ H(V|U) &\leq I(X_2; Y|X_1), \\ H(UV) &\leq I(X_1X_2; Y), \end{aligned}$$

where the information quantities are with respect to a distribution of the form $p(u, v)p(x_1)p(x_2)p(y|x_1, x_2)$. The codewords for the multiple access channel code in this scheme are generated independently according to the product distribution $p(x_1)p(x_2)$.

Cover, El Gamal, and Salehi (CES) demonstrated the failure of a separation strategy in the above scenario [3], in spite of a separation strategy being optimal for the case of a single sender and single receiver [25]. They found a simple example of a source with $p(0, 0) = p(0, 1) = p(1, 1) = 1/3$ and $p(1, 0) = 0$ and a channel $Y = X_1 + X_2$ such that the above information inequalities do not hold ($H(UV) = \log_2 3 = 1.58$ while $\max_{p(x_1)p(x_2)} I(X_1X_2; Y) = 1.5$), whereas the simple “joint” strategy of sending the source directly over the channel succeeds perfectly (set $X_1 = U$ and $X_2 = V$ so that the receiver can determine U and V from Y). More generally, the main result of their paper is a joint source-channel coding strategy to allow for the senders to transmit a source reliably over the channel, provided that the following information

inequalities hold:

$$\begin{aligned} H(U|V) &\leq I(X_1; Y|X_2VS), \\ H(V|U) &\leq I(X_2; Y|X_1US), \\ H(UV|W) &\leq I(X_1X_2; Y|WS), \\ H(UV) &\leq I(X_1X_2; Y), \end{aligned}$$

where the variable W is defined to be the “common part” of U and V (to be defined later), and the information quantities are with respect to a distribution $p(u, v)p(s)p(x_1|u, s)p(x_2|v, s)p(y|x_1, x_2)$. The advantage of the combined source-channel coding strategy is that the codebooks for the multiple access channel are now allowed to be correlated, because they are generated conditionally from the correlated source $p(u, v)$.

We can also study the combined source-channel coding problem in the context of quantum information theory. Quantum channels are models of communication which aim to fully represent the quantum degrees of freedom associated with a given communication medium [12, 23]. Understanding quantum degrees of freedom is of practical importance in the context of optical communications (for the case of bosonic channels). For example, it is well known that a quantum strategy at the receiver’s end of an optical channel can outperform a classical strategy such as homodyne, heterodyne, or direct detection [7, 8, 9, 10]. These latter results suggest that we should consider the extension of the classical information theoretic results to the quantum domain in order to study the source-channel problem.

The CES example above demonstrates that a separation strategy is not optimal in general for the case of a quantum multiple access channel [29], in contrast to a separation strategy being optimal for the single-sender single-receiver quantum channel [4]. (This result follows simply because every classical channel is a special type of quantum channel.) For these reasons, we feel that it is important to extend the CES theorem to the domain of quantum information theory.

The main accomplishment of the present paper is a generalization of the CES theorem to the case of quantum multiple access channels with classical inputs x_1 and x_2 and quantum outputs ρ_{x_1, x_2} . Our main result is the following theorem:

Theorem 1 *A source $(U^n, V^n) \sim \prod_{i=1}^n p(u_i, v_i)$ can be sent with arbitrarily small probability of error over a cq multiple access channel $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}, \mathcal{H}^B)$, with allowed codes $\{x_1^n(u^n), x_2^n(v^n)\}$ if there exist probability mass functions $p(s)$, $p(x_1|u, s)$, $p(x_2|v, s)$, such that the following information inequalities hold*

$$\begin{aligned} H(U|V) &\leq I(X_1; B|X_2VS), \\ H(V|U) &\leq I(X_2; B|X_1US), \\ H(UV|W) &\leq I(X_1X_2; B|WS), \\ H(UV) &\leq I(X_1X_2; B). \end{aligned}$$

The above information quantities are calculated with respect to the following classical-quantum state:

$$\sum_{u, v, w, s, x_1, x_2} p(u, v, w)p(s)p(x_1|u, s)p(x_2|v, s)|u, v, w, s, x_1, x_2\rangle\langle u, v, w, s, x_1, x_2|^{UVWSX_1X_2} \otimes \rho_{x_1, x_2}^B. \quad (1)$$

The distribution $p(u, v, w)$ is defined as

$$p(u, v, w) = p(u, v) \delta_{f(u), w} \delta_{g(v), w},$$

so that the random variable $W = f(U) = g(V)$ represents the common part of (U, V) .

Our proof of the above theorem is significantly different from the way that CES proved the classical version in Ref. [3] (and even different from the later simplification in Ref. [2]). First, we require the use of quantum typical projectors [12, 23, 27] (reviewed in an appendix), and furthermore, we exploit and extend

several recent advances made in the context of the quantum interference channel [24, 6] (later applied in other contexts such as the quantum broadcast channel [21], the quantum relay channel [22], and an entanglement-assisted quantum multiple access channel [30]). Even if one were to consider the case of our approach applied to a classical channel, our approach is different from that in Refs. [3, 2] (consider how our decomposition of the “error events” in (9) differs from the error event decomposition in Ref. [3]).

A quantum successive decoding strategy [29, 6] cannot achieve the region given in Theorem 1, demonstrating a clear advantage of the quantum simultaneous decoding technique [24, 6] in the setting of the quantum multiple access channel. We expect these simultaneous decoding techniques to extend to many other scenarios in network quantum information theory (indeed consider those already developed in Refs. [21, 22, 30]), in spite of the current limitation of the approach to decoding the transmissions of at most two senders.

In the next section, we describe the setting of the problem and define the information processing task that we are considering. Section 2 provides a detailed proof of Theorem 1, and we conclude with some final remarks and suggestions for open problems in Section 3.

1 Information Processing Task

Suppose that two IID information sources U_1, U_2, \dots, U_n and V_1, V_2, \dots, V_n with distribution

$$p(u^n, v^n) \equiv \prod_{i=1}^n p(u_i, v_i)$$

are available to two senders, respectively.

Suppose that it is possible to arrange the matrix for the joint distribution $p(u, v)$ into the following block-diagonal form

$$\bigoplus_{w=1}^k \begin{bmatrix} p(u_1^{(w)}, v_1^{(w)}) & \cdots & p(u_1^{(w)}, v_{m_w}^{(i)}) \\ \vdots & \ddots & \vdots \\ p(u_{l_w}^{(w)}, v_1^{(w)}) & \cdots & p(u_{l_w}^{(w)}, v_{m_w}^{(w)}) \end{bmatrix},$$

for some $w \in \{1, \dots, k\}$ and integers l_w and m_w . Then the common part [3, 5] of U and V is the random variable W that is equal to w if (u, v) is in block $w \in \{1, \dots, k\}$. Observe that it is possible to determine the common part from either U or V alone, simply by determining the block to which a realization u or v belongs. We can define this common part by finding the maximum integer k such that there exist functions f and g

$$\begin{aligned} f : U &\rightarrow \{1, \dots, k\}, \\ g : V &\rightarrow \{1, \dots, k\}, \end{aligned}$$

with $\Pr\{f(U) = w\} > 0$, $\Pr\{g(V) = w\} > 0$, for $w \in \{1, \dots, k\}$ with $\Pr\{f(U) = g(V)\} = 1$. We define $W = f(U) = g(V)$. One can then write the distribution of U , V , and W as

$$p(u, v, w) = p(u, v) \delta_{f(u), w} \delta_{g(v), w}.$$

A combined code for the source and the quantum multiple access channel consists of two encoding functions:

$$\begin{aligned} x_1^n : \mathcal{U}^n &\rightarrow \mathcal{X}_1^n, \\ x_2^n : \mathcal{V}^n &\rightarrow \mathcal{X}_2^n, \end{aligned}$$

that assign codewords suitable for input to the quantum multiple access channel for each of the possible outputs of the source. Upon input of these codewords $x_1^n(u^n)$ and $x_2^n(v^n)$ to the quantum multiple access channel, the quantum output for the receiver is

$$\rho_{x_1^n(u^n), x_2^n(v^n)} = \rho_{x_{11}(u^n), x_{21}(v^n)} \otimes \cdots \otimes \rho_{x_{1n}(u^n), x_{2n}(v^n)}.$$

The receiver performs a decoding positive operator-valued measure (POVM) $\{\Lambda_{u^n, v^n}\}$ whose elements Λ_{u^n, v^n} are positive and form a resolution of the identity:

$$\begin{aligned}\Lambda_{u^n, v^n} &\geq 0, \\ \sum_{(u^n, v^n) \in \mathcal{U}^n \times \mathcal{V}^n} \Lambda_{u^n, v^n} &= I^{\otimes n}.\end{aligned}$$

The end-to-end probability of error $P_e^{(n)}$ in this protocol is the probability that the source is decoded incorrectly by the receiver:

$$P_e^{(n)} \equiv \sum_{(u^n, v^n) \in \mathcal{U}^n \times \mathcal{V}^n} p(u^n, v^n) \text{Tr} \{ (I - \Lambda_{u^n, v^n}) \rho_{x_1^n(u^n), x_2^n(v^n)} \}.$$

We say that it is possible to transmit the source reliably if there exists a sequence of block codes $\{x_1^n(u^n), x_2^n(v^n)\}$ and a corresponding decoding POVM $\{\Lambda_{u^n, v^n}\}$ such that $P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$.

2 Achievability Proof

This section provides a proof of Theorem 1, by allowing for the generation of a code randomly and analyzing the expectation of the error probability under this random choice. The main steps in the proof are the random construction of the codebooks, the design of a decoding measurement for the receiver, and an error analysis demonstrating that the inequalities in Theorem 1 are a sufficient set of conditions for transmitting the correlated source reliably. The random construction of the codebooks is identical to that in the CES paper [3], but the difference for the quantum case lies in designing a collective decoding measurement for the receiver (this is always the difference between any protocol for classical communication over a classical channel and one for classical communication over a quantum channel). Our decoding POVM exploits and extends the quantum simultaneous decoding techniques from Refs. [24, 6], though the situation considered here is rather different from the one considered there. First, we demand that the receiver only decode source sequences that are jointly typical. It is natural that such an approach should perform well asymptotically, given that nearly all of the source probability concentrates on this set. Second, our decoding POVM has several conditionally typical projectors “sandwiched” together in a very particular order, such that our error analysis can proceed in bounding errors in terms of the information quantities appearing in Theorem 1. The decoding measurement is a square-root measurement [12, 23], though nothing would prevent us from employing a sequential, simultaneous decoder of the flavor in Ref. [24]. Our error analysis carefully employs indicator functions over the jointly typical set of source sequences (since we are decoding only over this set), and this approach allows us to introduce distributions that are needed for taking expectations of quantum codeword states. Finally, we decompose error terms in a way different from that in the CES paper [3], and in our opinion, this error decomposition appears to be a bit more natural than the error event decomposition appearing there.

Proof. We split the proof into several stages.

Code Generation. For every $w^n \in \mathcal{W}^n$, independently generate one s^n sequence according to

$$p(s^n) = \prod_{i=1}^n p(s_i).$$

Index them as $s^n(w^n)$ for all $w^n \in \mathcal{W}^n$.

For every $u^n \in \mathcal{U}^n$, compute the corresponding $w^n = f^n(u^n) \equiv f(u_1) \cdots f(u_n)$ and independently generate one x_1^n sequence according to

$$p(x_1^n | u^n, s^n(w^n)) = \prod_{i=1}^n p(x_{1i} | u_i, s_i(w^n)).$$

Index these x_1^n sequences according to $x_1^n(u^n, s^n(w^n))$.

Similarly, for every $v^n \in \mathcal{V}^n$, compute the corresponding $w^n = g^n(v^n) \equiv g(v_1) \cdots g(v_n)$ and independently generate one x_2^n sequence according to

$$p(x_2^n | v^n, s^n(w^n)) = \prod_{i=1}^n p(x_{2i} | v_i, s_i(w^n)).$$

Index these x_2^n sequences according to $x_2^n(v^n, s^n(w^n))$.

Encoding. Upon observing the output u^n of the source, Sender 1 computes $w^n = f^n(u^n)$ and transmits $x_1^n(u^n, s^n(w^n))$. Similarly, upon observing the output v^n of the source, Sender 2 computes $w^n = g^n(v^n)$ and transmits $x_2^n(v^n, s^n(w^n))$.

Decoding POVM. We now need to determine a decoding POVM that the receiver can perform to detect what codewords the senders transmitted. In this vein, we need to consider several typical projectors (see Appendix A) corresponding to several reduced states of (1). By considering the following information equalities, we can determine which reduced states we will require:

$$\begin{aligned} I(X_1; B | X_2 V S) &= H(B | X_2 V S) - H(B | X_1 X_2 V S) \\ &= H(B | X_2 V S) - H(B | X_1 X_2), \\ I(X_2; B | X_1 U S) &= H(B | X_1 U S) - H(B | X_1 X_2 U S) \\ &= H(B | X_1 U S) - H(B | X_1 X_2), \\ I(X_1 X_2; B | W S) &= H(B | W S) - H(B | X_1 X_2 W S) \\ &= H(B | W S) - H(B | X_1 X_2), \\ I(X_1 X_2; B) &= H(B) - H(B | X_1 X_2). \end{aligned}$$

Thus, it is clear that five different reduced states corresponding to the entropies $H(B | X_2 V S)$, $H(B | X_1 U S)$, $H(B | W S)$, $H(B)$, and $H(B | X_1 X_2)$ are important.

First, we consider the reduced state on $X_2 V S B$:

$$\sum_{v, s, x_2} p(v) p(s) p(x_2 | v, s) |v, s, x_2\rangle \langle v, s, x_2|^{V S X_2} \otimes \sigma_{v, s, x_2}^B,$$

where we define the states σ_{v, s, x_2}^B as

$$\begin{aligned} \sigma_{v, s, x_2}^B &\equiv \sum_{u, w, x_1} p(u, w | v) p(x_1 | u, s) \rho_{x_1, x_2}^B \\ &= \sum_{u, x_1} p(u | v) p(x_1 | u, s) \rho_{x_1, x_2}^B. \end{aligned} \tag{2}$$

The second equality follows from (13). Corresponding to a tensor product $\sigma_{v^n, s^n, x_2^n}^{B^n} \equiv \sigma_{v_1, s_1, x_{21}}^{B_1} \otimes \cdots \otimes \sigma_{v_n, s_n, x_{2n}}^{B_n}$ of these states is a conditionally typical projector $\Pi_{\sigma_{v^n, s^n, x_2^n}^{B^n}}$ which we abbreviate as $\Pi_{x_2^n(v^n, s^n)}$. The rank of each of these conditionally typical projectors is approximately equal to $2^{nH(B | X_2 V S)}$. Let $\rho_{x_2, s}^{(u)}$ denote the following state:

$$\rho_{x_2, s}^{(u)} \equiv \sum_{x_1} p(x_1 | u, s) \rho_{x_1, x_2}^B. \tag{3}$$

Next, we consider the reduced state on $X_1 U S B$:

$$\sum_{u, s, x_1} p(u) p(s) p(x_1 | u, s) |u, s, x_1\rangle \langle u, s, x_1|^{U S X_1} \otimes \omega_{u, s, x_1}^B,$$

where we define the states $\omega_{u,s,x_1}^B = \sum_{v,x_2} p(v|u) p(x_2|v,s) \rho_{x_1,x_2}^B$. Corresponding to a tensor product $\omega_{u^n,s^n,x_1^n}^{B^n}$ of these states is a conditionally typical projector $\Pi_{\omega_{u^n,s^n,x_1^n}^B}$ which we abbreviate as $\Pi_{x_1^n(u^n,s^n)}$. The rank of each of these conditionally typical projectors is approximately equal to $2^{nH(B|X_1US)}$. Let $\rho_{x_1,s}^{(v)}$ denote the following state:

$$\rho_{x_1,s}^{(v)} \equiv \sum_{x_2} p(x_2|v,s) \rho_{x_1,x_2}^B. \quad (4)$$

Third, we consider the reduced state on WSB :

$$\sum_{w,s} p(w) p(s) |w,s\rangle \langle w,s|^{WS} \otimes \tau_{w,s}^B,$$

where the states $\tau_{w,s}^B$ are defined as

$$\tau_{w,s}^B \equiv \sum_{x_1,x_2,u,v} p(u,v|w) p(x_1|u,s) p(x_2|v,s) \rho_{x_1,x_2}^B. \quad (5)$$

Corresponding to a tensor product $\tau_{w^n,s^n}^{B^n}$ of these states is a conditionally typical projector $\Pi_{\tau_{w^n,s^n}^B}$ which we abbreviate as $\Pi_{s^n(w^n)}$. The rank of each of these conditionally typical projectors is approximately equal to $2^{nH(B|WS)}$.

Fourth, we have the state $\bar{\rho}^B$ obtained by tracing over every system except B :

$$\bar{\rho}^B \equiv \sum_{u,v,w,s,x_1,x_2} p(u,v,w) p(s) p(x_1|u,s) p(x_2|v,s) \rho_{x_1,x_2}^B.$$

To a tensor power $(\bar{\rho}^B)^{\otimes n}$ of this state corresponds a typical projector $\Pi_{\bar{\rho}^{\otimes n}}$, which we abbreviate as just Π . The rank of this projector is approximately equal to $2^{nH(B)}$.

Finally, we define the tensor product states $\rho_{x_1^n,x_2^n}^{B^n}$, which correspond to the outputs of the channel when x_1^n and x_2^n are input. To these states corresponds a conditionally typical projector $\Pi_{\rho_{x_1^n,x_2^n}^B}$, which we abbreviate as $\Pi_{x_1^n(x_2^n)}$. Its rank is approximately equal to $2^{nH(B|X_1X_2)}$.

We can now construct the decoding POVM for the receiver. For every $(u^n, v^n, w^n) \in \mathcal{T}^n$ (with \mathcal{T}^n the joint weakly typical set defined in Appendix B), consider the following operators:

$$\Gamma_{u^n,v^n,w^n} \equiv \Pi \Pi_{s^n(w^n)} \Pi_{x_1^n(u^n,s^n)} \Pi_{x_1^n(u^n,s^n),x_2^n(v^n,s^n)} \Pi_{x_1^n(u^n,s^n)} \Pi_{s^n(w^n)} \Pi. \quad (6)$$

Note that we define these operators only over $(u^n, v^n, w^n) \in \mathcal{T}^n$ and set them equal to zero otherwise. The measurement for the receiver is a “square-root” measurement of the following form:

$$\Lambda_{u^n,v^n,w^n} \equiv \left(\sum_{(u'^n,v'^n,w'^n) \in \mathcal{T}^n} \Gamma_{u'^n,v'^n,w'^n} \right)^{-1/2} \Gamma_{u^n,v^n,w^n} \left(\sum_{(u'^n,v'^n,w'^n) \in \mathcal{T}^n} \Gamma_{u'^n,v'^n,w'^n} \right)^{-1/2}.$$

Error Analysis. The decoding error is defined as follows:

$$\sum_{(u^n,v^n) \in \mathcal{U}^n \times \mathcal{V}^n} p(u^n, v^n) \Pr \{ \text{error made at decoder} \mid (u^n, v^n) \text{ is the output of the source} \}.$$

For our case, this reduces to

$$\begin{aligned} & \sum_{u^n,v^n} p(u^n, v^n) \text{Tr} \{ (I - \Lambda_{u^n,v^n,w^n}) \rho_{x_1^n(u^n,s^n),x_2^n(v^n,s^n)} \} \\ &= \sum_{u^n,v^n,w^n} p(u^n, v^n, w^n) \text{Tr} \{ (I - \Lambda_{u^n,v^n,w^n}) \rho_{x_1^n(u^n,s^n),x_2^n(v^n,s^n)} \}. \end{aligned}$$

We also allow for an expectation over the random choice of code. We will introduce this when needed in order to simplify the error analysis (at the end, we will determine that the expectation is not necessary—that there exists some code by a derandomization argument):

$$\sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ (I - \Lambda_{u^n, v^n, w^n}) \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\}.$$

Our first “move” is to smooth the state by the projector $\Pi_{x_2^n(v^n, S^n)}$ (this is effectively “smoothing” the channel). That is, we can bound the error from above as

$$\begin{aligned} & \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \text{Tr} \left\{ (I - \Lambda_{u^n, v^n, w^n}) \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \\ & + \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\| \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} - \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\|_1. \end{aligned} \quad (7)$$

By introducing the expectation over the random choice of code $\mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n}$, we can bound the second term from above by $2\sqrt{\epsilon}$, by applying the Gentle Operator Lemma for ensembles (Appendix C) and the properties of quantum typicality:

$$\begin{aligned} & \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \\ & \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \left\| \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} - \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\|_1 \right\} \leq 2\sqrt{\epsilon} \end{aligned}$$

We now consider the first term in (7) and make the abbreviation

$$\rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \equiv \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)}.$$

This term naturally splits up into two parts according to the typicality condition $(u^n, v^n, w^n) \in \mathcal{T}^n$:

$$\begin{aligned} & \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\{ \text{Tr} \left\{ (I - \Lambda_{u^n, v^n, w^n}) \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} [\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) + \mathcal{I}((u^n, v^n, w^n) \notin \mathcal{T}^n)] \\ & = \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\{ \text{Tr} \left\{ (I - \Lambda_{u^n, v^n, w^n}) \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) + \\ & \quad \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\{ \text{Tr} \left\{ \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \mathcal{I}((u^n, v^n, w^n) \notin \mathcal{T}^n) \end{aligned} \quad (8)$$

where in the last line we used that $\Lambda_{u^n, v^n, w^n} = 0$ whenever $(u^n, v^n, w^n) \notin \mathcal{T}^n$. (We have also introduced an indicator function $\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n)$ which is equal to one if $(u^n, v^n, w^n) \in \mathcal{T}^n$ equal to zero otherwise.) We now apply the well-known Hayashi-Nagaoka operator inequality (Lemma 2 of Ref. [11]) to the term in the middle line, giving the upper bound

$$\begin{aligned} & 2 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\{ \text{Tr} \left\{ (I - \Gamma_{u^n, v^n, w^n}) \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) + \\ & 4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \\ & \quad \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n) \end{aligned}$$

Doubling the term in the last line of (8) and adding it to the first above gives the following upper bound

$$\begin{aligned}
& 2 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \left\{ \text{Tr} \left\{ (I - \Gamma_{u^n, v^n, w^n}) \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} + \\
& 4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \\
& \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n)
\end{aligned}$$

We bound the first term, by introducing the expectation $\mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n}$ and considering that

$$\begin{aligned}
& \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u^n, v^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\
& = \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \text{Tr} \left\{ \Gamma_{U^n, V^n, X^n} \rho'_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \right\} \right\} \\
& = \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(W^n)} \Pi_{X_1^n(U^n, S^n)} \Pi_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi_{X_1^n(U^n, S^n)} \Pi_{S^n(W^n)}}{\Pi \Pi_{X_2^n(V^n, S^n)} \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi_{X_2^n(V^n, S^n)}} \times \right\} \right\} \\
& \geq \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \text{Tr} \left\{ \Pi_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \right\} \right\} \\
& \quad - \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \left\| \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} - \Pi_{X_2^n(V^n, S^n)} \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi_{X_2^n(V^n, S^n)} \right\|_1 \right\} \\
& \quad - \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \left\| \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} - \Pi \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi \right\|_1 \right\} \\
& \quad - \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \left\| \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} - \Pi_{S^n(W^n)} \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi_{S^n(W^n)} \right\|_1 \right\} \\
& \quad - \mathbb{E}_{U^n V^n W^n S^n X_1^n X_2^n} \left\{ \left\| \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} - \Pi_{X_1^n(U^n, S^n)} \rho_{X_1^n(U^n, S^n), X_2^n(V^n, S^n)} \Pi_{X_1^n(U^n, S^n)} \right\|_1 \right\} \\
& \geq 1 - \epsilon - 8\sqrt{\epsilon},
\end{aligned}$$

where we have used the trace inequality $\text{Tr}\{\Lambda\rho\} \geq \text{Tr}\{\Lambda\sigma\} - \|\rho - \sigma\|_1$, properties of quantum typicality, and the Gentle Operator Lemma for ensembles. Thus, we obtain the following upper bound on the error:

$$\begin{aligned}
& 2(\epsilon + 8\sqrt{\epsilon}) + 4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \\
& \sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n).
\end{aligned}$$

The sum $\sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} (\cdot)$ can split up in the following seven different ways:

$$\sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} (\cdot) = \sum_{\substack{u'^n \neq u^n, \\ v'^n = v^n, \\ w'^n = w^n}} (\cdot) + \sum_{\substack{u'^n = u^n, \\ v'^n \neq v^n, \\ w'^n = w^n}} (\cdot) + \sum_{\substack{u'^n = u^n, \\ v'^n = v^n, \\ w'^n \neq w^n}} (\cdot) + \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n, \\ w'^n = w^n}} (\cdot) + \sum_{\substack{u'^n \neq u^n, \\ v'^n = v^n, \\ w'^n \neq w^n}} (\cdot) + \sum_{\substack{u'^n = u^n, \\ v'^n \neq v^n, \\ w'^n \neq w^n}} (\cdot) + \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n, \\ w'^n \neq w^n}} (\cdot).$$

Though, considering that $u'^n = u^n$ implies that $f^n(u'^n) = f^n(u^n)$ and thus that $w'^n = w^n$, the terms in the following sums do not occur:

$$\sum_{\substack{u'^n = u^n, \\ v'^n = v^n, \\ w'^n \neq w^n}} (\cdot) + \sum_{\substack{u'^n = u^n, \\ v'^n \neq v^n, \\ w'^n \neq w^n}} (\cdot).$$

Similarly, we have that $v'^n = v^n$ implies that $g^n(v'^n) = g^n(v^n)$ and thus that $w'^n = w^n$, and so the terms in the following sum do not occur:

$$\sum_{\substack{u'^n \neq u^n, \\ v'^n = v^n, \\ w'^n \neq w^n}} (\cdot).$$

So this leaves us with the following decomposition of the overall sum into four different sums:

$$\sum_{(u'^n, v'^n, w'^n) \neq (u^n, v^n, w^n)} (\cdot) = \underbrace{\sum_{\substack{u'^n \neq u^n, \\ v'^n = v^n, \\ w'^n = w^n}} (\cdot)}_{\alpha} + \underbrace{\sum_{\substack{u'^n = u^n, \\ v'^n \neq v^n, \\ w'^n = w^n}} (\cdot)}_{\beta} + \underbrace{\sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n, \\ w'^n = w^n}} (\cdot)}_{\gamma} + \underbrace{\sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n, \\ w'^n \neq w^n}} (\cdot)}_{\Delta}, \quad (9)$$

where we label the four different error terms as α , β , γ , and Δ .

We now analyze each of these sums individually and introduce the expectation $\mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n}$, which corresponds to the expectations over the random choice of codebook.

Error α . Consider the error term from the first sum:

$$4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n).$$

Let us focus on bounding the middle part of the above expression:

$$\begin{aligned} & \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\ &= \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \\ &= \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \mathbb{E}_{X_1^n | u^n S^n} \left\{ \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \\ &= \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \rho_{X_2^n(v^n, S^n)}^{(u^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \end{aligned}$$

The first equality follows from the definition of $\rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}$. The second equality follows because $u'^n \neq u^n$ and from the way that the code was chosen randomly. The third equality is from the definition in (3):

$$\mathbb{E}_{X_1^n | u^n S^n} \left\{ \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} = \rho_{X_2^n(v^n, S^n)}^{(u^n)}.$$

At this point, we substitute back into our earlier expression and continue upper bounding:

$$\sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \rho_{X_2^n(v^n, S^n)}^{(u^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \times \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n)$$

Consider the following inequality:

$$\begin{aligned} \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) &= \sum_{u^n, v^n, w^n} p(u^n | v^n) p(v^n) \delta_{w^n, f^n(u^n)} \delta_{w^n, g^n(v^n)} \\ &\leq \sum_{u^n, v^n, w^n} p(u^n | v^n) p(v^n) \delta_{w^n, g^n(v^n)} \end{aligned}$$

Substituting and distributing gives us the upper bound:

$$\begin{aligned}
&\leq \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \times \\
&\quad \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \left(\sum_{u^n} p(u^n | v^n) \rho_{X_2^n(v^n, S^n)}^{(u^n)} \right) \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \times \\
&\quad \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n). \\
&\leq \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \times \\
&\quad \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \left(\sum_{u^n} p(u^n | v^n) \rho_{X_2^n(v^n, S^n)}^{(u^n)} \right) \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \times \\
&\quad \mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n),
\end{aligned}$$

where the second inequality follows from dropping the indicator function $\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n)$. Focusing on the expression in the middle (and omitting the indicator function for the moment), we have

$$\begin{aligned}
&\sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \left(\sum_{u^n} p(u^n | v^n) \rho_{X_2^n(v^n, S^n)}^{(u^n)} \right) \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \\
&= \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \sigma_{X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \\
&\leq 2^{-n[H(B|X_2VS)-\delta]} \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v^n, w^n} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \\
&= 2^{-n[H(B|X_2VS)-\delta]} \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v^n, S^n)} \times}{\Pi_{X_1^n(u'^n, S^n)} \Pi_{S^n(w^n)} \Pi_{X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{-n[H(B|X_2VS)-\delta]} \sum_{u'^n \neq u^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Pi_{X_1^n(u'^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\
&\leq 2^{-n[H(B|X_2VS)-\delta]} 2^{n[H(B|X_1X_2)+\delta]} \sum_{u'^n \neq u^n} 1
\end{aligned}$$

The first equality follows from the definition in (2):

$$\sigma_{X_2^n(v^n, S^n)} = \sum_{u^n} p(u^n | v^n) \rho_{X_2^n(v^n, S^n)}^{(u^n)}.$$

The first inequality follows from the property (12) of conditionally typical projectors, applied to the state $\sigma_{X_2^n(v^n, S^n)}$ and its corresponding typical projector $\Pi_{X_2^n(v^n, S^n)}$. The second equality follows from the definition (6) of $\Gamma_{u'^n, v^n, w^n}$. The second inequality is from cyclicity of trace and the fact that $\Pi, \Pi_{S^n(w^n)}, \Pi_{X_1^n(u'^n, S^n)} \leq I$. The final inequality is from the property (11) of typical projectors (for the projector $\Pi_{X_1^n(u'^n, S^n), X_2^n(v^n, S^n)}$). Substituting back in, we obtain the upper bound

$$\leq 2^{-n[H(B|X_2VS)-\delta]} 2^{n[H(B|X_1X_2)+\delta]} \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \sum_{u'^n \neq u^n} \mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n),$$

which we can further upper bound as

$$\begin{aligned}
&\leq 2^{-n[H(B|X_2VS)-\delta]} 2^{n[H(B|X_1X_2)+\delta]} \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \sum_{u'^n \neq u^n} p(u'^n | v^n, w^n) 2^{n[H(U|VW)+\delta']} \\
&\leq 2^{-n[H(B|X_2VS)-\delta]} 2^{n[H(B|X_1X_2)+\delta]} 2^{n[H(U|VW)+\delta']} \\
&= 2^{-n[I(X_1;B|X_2VS)-H(U|V)-2\delta-\delta']},
\end{aligned}$$

where we exploited the fact that

$$\mathcal{I}((u'^n, v^n, w^n) \in \mathcal{T}^n) \leq p(u'^n | v^n, w^n) 2^{n[H(U|VW)+\delta']}.$$

(See Appendix B for a discussion of this fact.) Substituting this back into our original expression, we have the bound:

$$\leq 4 \cdot 2^{-n[I(X_1;B|X_2VS)-H(UW|V)-2\delta-\delta']}.$$

Error β . We now handle the error from the second sum:

$$\begin{aligned}
&4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \\
&\quad \sum_{v'^n \neq v^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u^n, v'^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \\
&\quad \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u^n, v'^n, w^n) \in \mathcal{T}^n).
\end{aligned}$$

Let us again focus on bounding the middle part:

$$\sum_{v'^n \neq v^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u^n, v'^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\}$$

Expanding $\mathbb{E}_{X_2^n | v'^n S^n} \{ \Gamma_{u^n, v'^n, w^n} \}$ as

$$\begin{aligned}
&\mathbb{E}_{X_2^n | v'^n S^n} \{ \Gamma_{u^n, v'^n, w^n} \} \\
&= \mathbb{E}_{X_2^n | v'^n S^n} \{ \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \Pi_{X_1^n(u^n, S^n), X_2^n(v'^n, S^n)} \Pi_{X_1^n(u^n, S^n)} \Pi_{S^n(w^n)} \Pi \} \\
&= \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \mathbb{E}_{X_2^n | v'^n S^n} \{ \Pi_{X_1^n(u^n, S^n), X_2^n(v'^n, S^n)} \} \Pi_{X_1^n(u^n, S^n)} \Pi_{S^n(w^n)} \Pi \\
&\leq 2^{n[H(B|X_1X_2)+\delta]} \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \mathbb{E}_{X_2^n | v'^n S^n} \{ \rho_{X_1^n(u^n, S^n), X_2^n(v'^n, S^n)} \} \Pi_{X_1^n(u^n, S^n)} \Pi_{S^n(w^n)} \Pi \\
&= 2^{n[H(B|X_1X_2)+\delta]} \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \rho_{X_1^n(u^n, S^n)}^{(v'^n)} \Pi_{X_1^n(u^n, S^n)} \Pi_{S^n(w^n)} \Pi.
\end{aligned}$$

The first equality is a definition. The second equality follows because $v'^n \neq v^n$ for terms in the sum and from the way that the code was chosen. The first inequality is the “projector trick” discussed in Appendix A. The final equality is from the definition in (4). Substituting back in gives the upper bound

$$\begin{aligned}
&\leq 2^{n[H(B|X_1X_2)+\delta]} \sum_{v'^n \neq v^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \begin{array}{c} \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \rho_{X_1^n(u^n, S^n)}^{(v'^n)} \Pi_{X_1^n(u^n, S^n)} \times \\ \Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \end{array} \right\} \right\} \times \\
&\quad \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u^n, v'^n, w^n) \in \mathcal{T}^n)
\end{aligned}$$

We now note the upper bounds

$$\begin{aligned}
&\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u^n, v'^n, w^n) \in \mathcal{T}^n) \leq \mathcal{I}((u^n, v'^n) \in \mathcal{T}^n) \\
&\leq p(v'^n | u^n) 2^{n[H(V|U)+\delta']}
\end{aligned}$$

which upon substitution and distribution give the upper bound

$$\begin{aligned}
&\leq 2^{n[H(B|X_1X_2)+H(V|U)+\delta+\delta']} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \times \left(\sum_{v'^n} p(v'^n|u^n) \rho_{X_1^n(u^n, S^n)}^{(v'^n)} \right) \Pi_{X_1^n(u^n, S^n)} \times}{\Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{n[H(B|X_1X_2)+\delta]} 2^{n[H(V|U)+\delta']} 2^{-n[H(B|X_1US)-\delta]} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n)} \times}{\Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{n[H(B|X_1X_2)+\delta]} 2^{n[H(V|U)+\delta']} 2^{-n[H(B|X_1US)-\delta]} \\
&= 2^{-n[I(X_2;B|X_1US)-H(V|U)-\delta'-2\delta]}
\end{aligned}$$

Substituting back in then gives the bound:

$$\begin{aligned}
&4 \cdot 2^{-n[I(X_2;B|X_1US)-H(V|U)-\delta'-2\delta]} \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \\
&\leq 4 \cdot 2^{-n[I(X_2;B|X_1US)-H(V|U)-\delta'-2\delta]}.
\end{aligned}$$

Error γ . We now handle the error from the third sum (this error corresponds to the common part w^n being decoded correctly, but u^n and v^n both being decoded incorrectly):

$$\begin{aligned}
&4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \\
&\quad \sum_{u'^n \neq u^n, v'^n \neq v^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \\
&\quad \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w^n) \in \mathcal{T}^n).
\end{aligned}$$

Again consider the middle part:

$$\begin{aligned}
&\sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n}} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\
&= \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n}} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \times}{\Pi_{X_1^n(u'^n, S^n)} \Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{n[H(B|X_1X_2)+\delta]} \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n}} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \rho_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \times}{\Pi_{X_1^n(u'^n, S^n)} \Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\}
\end{aligned}$$

Here, we have again used the “projector trick inequality”:

$$\Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \leq 2^{n[H(B|X_1X_2)+\delta]} \rho_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)}.$$

Now we introduce the expectations over the random choice of codewords. Since $v'^n \neq v^n$, this means that there are two independent random codewords involved, namely $X_2^n(v^n, S^n(w^n))$ and $X_2^n(v'^n, S^n(w^n))$. We take the expectation $\mathbb{E}_{X_2^n|v^n S^n}$ inside the trace and obtain

$$2^{n[H(B|X_1X_2)+\delta]} \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n}} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n|u^n S^n, X_2^n|v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \Pi_{X_1^n(u'^n, S^n)} \times}{\Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\}$$

We will also exploit the following “indicator function trick” in order to bound the full error event probability:

$$\begin{aligned}
\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w^n) \in \mathcal{T}^n) &\leq \mathcal{I}((u'^n, v'^n, w^n) \in \mathcal{T}^n) \\
&\leq 2^{n[H(UV|W)+\delta']} p(u'^n, v'^n | w^n) \\
&= 2^{n[H(UV|W)+\delta']} p(u'^n | w^n) p(v'^n | u'^n, w^n) \\
&= 2^{n[H(UV|W)+\delta']} p(u'^n | w^n) p(v'^n | u'^n),
\end{aligned}$$

where the last equality holds because $p(v'^n | u'^n, w^n) = p(v'^n | u'^n)$ whenever $w^n = f(u^n) = g(v^n) = f(u'^n) = g(v'^n)$ (which we know is true for terms in this third sum—otherwise, the delta function from $p(u^n, v^n, w^n)$ ensures that the whole expression vanishes). Combining the results of the projector trick inequality and the indicator function trick, we then get the upper bound

$$\begin{aligned}
&\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} \sum_{\substack{u'^n \neq u^n, \\ v'^n \neq v^n}} p(u'^n | w^n) p(v'^n | u'^n) \times \\
&\quad \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \Pi_{X_1^n(u'^n, S^n)} \times}{\Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} \sum_{u'^n, v'^n} p(u'^n | w^n) \times \\
&\quad \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u'^n, S^n)} \left(\sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right) \times}{\Pi_{X_1^n(u'^n, S^n)} \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\}
\end{aligned}$$

Continuing, we have

$$\begin{aligned}
&\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} \sum_{u'^n} p(u'^n | w^n) \times \\
&\quad \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Pi \Pi_{S^n(w^n)} \left(\sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right) \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\
&= 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} \times \\
&\quad \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \left(\sum_{u'^n, v'^n} p(u'^n | w^n) \mathbb{E}_{X_1^n | u'^n S^n} \left\{ p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right\} \right) \Pi_{S^n(w^n)} \times}{\Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&= 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \frac{\Pi \Pi_{S^n(w^n)} \tau_{S^n(w^n)} \Pi_{S^n(w^n)} \times}{\Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}} \right\} \right\} \\
&\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} 2^{-n[H(B|WS)-\delta]} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n S^n, X_2^n | v^n S^n} \left\{ \text{Tr} \left\{ \Pi \Pi_{S^n(w^n)} \Pi_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \\
&\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UV|W)+\delta']} 2^{-n[H(B|WS)-\delta]} \\
&= 2^{-n[I(X_1 X_2; B|WS) - H(UV|W) - 2\delta - \delta']}.
\end{aligned}$$

The first inequality follows from the fact that

$$\begin{aligned}
& \Pi_{X_1^n(u'^n, S^n)} \left(\sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right) \Pi_{X_1^n(u'^n, S^n)} \\
&= \left(\sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right)^{1/2} \Pi_{X_1^n(u'^n, S^n)} \left(\sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right)^{1/2} \\
&\leq \sum_{v'^n} p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)}.
\end{aligned}$$

The first equality follows from bringing the sum $\sum_{u'^n} p(u'^n | w^n)$ and expectation $\mathbb{E}_{X_1^n | u'^n, S^n}$ inside the trace. The second equality follows from the definition in (5):

$$\tau_{S^n(w^n)} \equiv \sum_{u'^n, v'^n} p(u'^n | w^n) \mathbb{E}_{X_1^n | u'^n, S^n} \left\{ p(v'^n | u'^n) \rho_{X_1^n(u'^n, S^n)}^{(v'^n)} \right\}.$$

The second inequality follows from the property (12) of conditionally typical projectors:

$$\Pi_{S^n(w^n)} \tau_{S^n(w^n)} \Pi_{S^n(w^n)} \leq 2^{-n[H(B|WS)-\delta]} \Pi_{S^n(w^n)}.$$

The final inequality follows from the bound:

$$\text{Tr} \left\{ \Pi \Pi_{S^n(w^n)} \Pi \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \leq 1.$$

Error Δ . Finally, we upper bound the error from the last sum, which corresponds to all components of the source being erroneously decoded:

$$\begin{aligned}
& 4 \sum_{u^n, v^n, w^n} p(u^n, v^n, w^n) \times \\
& \sum_{u'^n \neq u^n, v'^n \neq v^n, w'^n \neq w^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n, S^n, X_2^n | v^n, S^n} \left\{ \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \rho'_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right\} \right\} \times \\
& \mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n) \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n). \\
& \leq 4 \sum_{u'^n, v'^n, w'^n} \sum_{u^n, v^n, w^n} p(v^n) p(u^n, w^n | v^n) \times \\
& \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \mathbb{E}_{S^n} \mathbb{E}_{X_1^n | u^n, S^n, X_2^n | v^n, S^n} \left\{ \Pi_{X_2^n(v^n, S^n)} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \times \\
& \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n),
\end{aligned}$$

where the inequality follows from dropping the indicator function $\mathcal{I}((u^n, v^n, w^n) \in \mathcal{T}^n)$. Considering that

$$\begin{aligned}
p(u^n, w^n | v^n) &= p(u^n | v^n) \delta_{w^n, f^n(u^n)} \delta_{w^n, g^n(v^n)} \\
&\leq p(u^n | v^n) \delta_{w^n, g^n(v^n)}
\end{aligned}$$

we have the upper bound:

$$\begin{aligned}
& \leq 4 \sum_{u'^n, v'^n, w'^n} \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \times \\
& \text{Tr} \left\{ \Gamma_{u'^n, v'^n, w'^n} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n | v^n, S^n} \left\{ \Pi_{X_2^n(v^n, S^n)} \left(\sum_{u^n} p(u^n | v^n) \mathbb{E}_{X_1^n | u^n, S^n} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right) \Pi_{X_2^n(v^n, S^n)} \right\} \right\} \times \\
& \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n).
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \sum_{u'^n, v'^n, w'^n} \sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \times \\
&\quad \text{Tr} \left\{ \frac{\Pi \Pi^{S^n(w'^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi^{S^n(w'^n)} \Pi}{\mathbb{E}_{S^n} \mathbb{E}_{X_2^n|v^n S^n} \left\{ \left(\sum_{u^n} p(u^n|v^n) \mathbb{E}_{X_1^n|u^n S^n} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right) \right\}} \right\} \times \\
&\quad \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n).
\end{aligned}$$

where the second inequality follows from the fact that $\sum_{u^n} p(u^n|v^n) \mathbb{E}_{X_1^n|u^n S^n} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}$ and $\Pi_{X_2^n(v^n, S^n)}$ commute. Continuing, we have

$$\begin{aligned}
&= 4 \sum_{u'^n, v'^n, w'^n} \text{Tr} \left\{ \frac{\Pi \Pi^{S^n(w'^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi^{S^n(w'^n)} \Pi}{\left(\sum_{v^n, w^n} p(v^n) \delta_{w^n, g^n(v^n)} \mathbb{E}_{S^n} \mathbb{E}_{X_2^n|v^n S^n} \left\{ \left(\sum_{u^n} p(u^n|v^n) \mathbb{E}_{X_1^n|u^n S^n} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)} \right) \right\} \right)} \right\} \times \\
&\quad \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n). \\
&= 4 \sum_{u'^n, v'^n, w'^n} \text{Tr} \left\{ \Pi^{S^n(w'^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi^{S^n(w'^n)} \Pi \bar{\rho}^{\otimes n} \Pi \right\} \times \\
&\quad \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n) \\
&\leq 4 2^{-n[H(B)-\delta]} \sum_{u'^n, v'^n, w'^n} \text{Tr} \left\{ \Pi^{S^n(w'^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)} \Pi_{X_1^n(u'^n, S^n)} \Pi^{S^n(w'^n)} \Pi \right\} \times \\
&\quad \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n) \\
&\leq 4 2^{-n[H(B)-\delta]} \sum_{u'^n, v'^n, w'^n} 2^{n[H(B|X_1 X_2)+\delta]} \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n) \\
&= 4 2^{-n[H(B)-\delta]} 2^{n[H(B|X_1 X_2)+\delta]} \sum_{u'^n, v'^n, w'^n} \mathcal{I}((u'^n, v'^n, w'^n) \in \mathcal{T}^n) \\
&\leq 4 2^{-n[H(B)-\delta]} 2^{n[H(B|X_1 X_2)+\delta]} 2^{n[H(UVW)+\delta]} \\
&= 4 2^{-n[I(X_1 X_2; B) - H(UVW) - 3\delta]}.
\end{aligned}$$

The first equality holds from the definition of $\bar{\rho}$ as the average output state

$$\bar{\rho}^{\otimes n} = \sum_{u^n, v^n} p(u^n, v^n) \mathbb{E}_{S^n} \mathbb{E}_{X_2^n|v^n S^n} \mathbb{E}_{X_1^n|u^n S^n} \rho_{X_1^n(u^n, S^n), X_2^n(v^n, S^n)}.$$

The first inequality follows from (10). The second inequality follows because $\Pi_{X_1^n(u'^n, S^n)}$, $\Pi^{S^n(w'^n)}$ and Π are all less than the identity and the property (11) of the projector $\Pi_{X_1^n(u'^n, S^n), X_2^n(v'^n, S^n)}$.

Thus, it is clear that as long as the information inequalities in the statement of Theorem 1 hold, then the expectation of the error probability can be made arbitrarily small as n becomes large. This implies the existence of a particular code which accomplishes the desired information processing task. ■

3 Conclusion

We have proved a quantum generalization of the Cover-El Gamal-Salehi theorem, regarding joint source-channel encoding for transmission of a classical correlated source over a multiple access channel. The theorem should be useful in applications concerning communication over bosonic quantum multiple access channels [32]. The main tools needed to prove this theorem are an extension of those developed in the context of the quantum interference channel [24, 6], combined with some further analysis using the “indicator function trick.”

Future work might pursue extensions of Theorem 1 to other interesting scenarios:

1. Joint source-channel coding (JSCC) with extra side information [20].
2. JSCC where the criterion is replaced by lossy transmission in the rate-distortion sense [15].
3. The goal of the protocol is computation over the multiple access channel [17].
4. JSCC in analog-digital hybrid coding scenarios [16].

Another very interesting pursuit would be the “fully quantum” generalization of this problem, where the goal would be to transmit a correlated quantum source over a quantum multiple access channel. The decoupling techniques from Ref. [13] should be useful in this context, and one would expect a nice extension of the theorem from Refs. [13, 31].

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A Typical Sequences and Typical Subspaces

Consider a density operator ρ with the following spectral decomposition:

$$\rho = \sum_x p_X(x) |x\rangle \langle x|.$$

The weakly typical subspace is defined as the span of all vectors such that the sample entropy $\overline{H}(x^n)$ of their classical label is close to the true entropy $H(X)$ of the distribution $p_X(x)$ [18, 27]:

$$T_\delta^{X^n} \equiv \text{span} \{ |x^n\rangle : |\overline{H}(x^n) - H(X)| \leq \delta \},$$

where

$$\begin{aligned} \overline{H}(x^n) &\equiv -\frac{1}{n} \log(p_{X^n}(x^n)), \\ H(X) &\equiv -\sum_x p_X(x) \log p_X(x). \end{aligned}$$

The projector $\Pi_{\rho,\delta}^n$ onto the typical subspace of ρ is defined as

$$\Pi_{\rho,\delta}^n \equiv \sum_{x^n \in T_\delta^{X^n}} |x^n\rangle \langle x^n|,$$

where we have “overloaded” the symbol $T_\delta^{X^n}$ to refer also to the set of δ -typical sequences:

$$T_\delta^{X^n} \equiv \{x^n : |\overline{H}(x^n) - H(X)| \leq \delta\}.$$

The three important properties of the typical projector are as follows:

$$\begin{aligned} \text{Tr} \{ \Pi_{\rho,\delta}^n \rho^{\otimes n} \} &\geq 1 - \epsilon, \\ \text{Tr} \{ \Pi_{\rho,\delta}^n \} &\leq 2^{n[H(X)+\delta]}, \\ 2^{-n[H(X)+\delta]} \Pi_{\rho,\delta}^n &\leq \Pi_{\rho,\delta}^n \rho^{\otimes n} \Pi_{\rho,\delta}^n \leq 2^{-n[H(X)-\delta]} \Pi_{\rho,\delta}^n, \end{aligned} \tag{10}$$

where the first property holds for arbitrary $\epsilon, \delta > 0$ and sufficiently large n . Consider an ensemble $\{p_X(x), \rho_x\}_{x \in \mathcal{X}}$ of states. Suppose that each state ρ_x has the following spectral decomposition:

$$\rho_x = \sum_y p_{Y|X}(y|x) |y_x\rangle \langle y_x|.$$

Consider a density operator ρ_{x^n} which is conditional on a classical sequence $x^n \equiv x_1 \cdots x_n$:

$$\rho_{x^n} \equiv \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}.$$

We define the weak conditionally typical subspace as the span of vectors (conditional on the sequence x^n) such that the sample conditional entropy $\overline{H}(y^n|x^n)$ of their classical labels is close to the true conditional entropy $H(Y|X)$ of the distribution $p_{Y|X}(y|x)p_X(x)$ [18, 27]:

$$T_\delta^{Y^n|x^n} \equiv \text{span} \{ |y_{x^n}^n\rangle : |\overline{H}(y^n|x^n) - H(Y|X)| \leq \delta \},$$

where

$$\begin{aligned} \overline{H}(y^n|x^n) &\equiv -\frac{1}{n} \log(p_{Y^n|X^n}(y^n|x^n)), \\ H(Y|X) &\equiv -\sum_x p_X(x) \sum_y p_{Y|X}(y|x) \log p_{Y|X}(y|x). \end{aligned}$$

The projector $\Pi_{\rho_{x^n}, \delta}$ onto the weak conditionally typical subspace of ρ_{x^n} is as follows:

$$\Pi_{\rho_{x^n}, \delta} \equiv \sum_{y^n \in T_\delta^{Y^n|x^n}} |y_{x^n}^n\rangle \langle y_{x^n}^n|,$$

where we have again overloaded the symbol $T_\delta^{Y^n|x^n}$ to refer to the set of weak conditionally typical sequences:

$$T_\delta^{Y^n|x^n} \equiv \{y^n : |\overline{H}(y^n|x^n) - H(Y|X)| \leq \delta\}.$$

The three important properties of the weak conditionally typical projector are as follows:

$$\begin{aligned} \mathbb{E}_{X^n} \{ \text{Tr} \{ \Pi_{\rho_{X^n}, \delta} \rho_{X^n} \} \} &\geq 1 - \epsilon, \\ \text{Tr} \{ \Pi_{\rho_{x^n}, \delta} \} &\leq 2^{n[H(Y|X) + \delta]}, \end{aligned} \tag{11}$$

$$2^{-n[H(Y|X) + \delta]} \Pi_{\rho_{x^n}, \delta} \leq \Pi_{\rho_{x^n}, \delta} \rho_{x^n} \Pi_{\rho_{x^n}, \delta} \leq 2^{-n[H(Y|X) - \delta]} \Pi_{\rho_{x^n}, \delta}, \tag{12}$$

where the first property holds for arbitrary $\epsilon, \delta > 0$ and sufficiently large n , and the expectation is with respect to the distribution $p_{X^n}(x^n)$.

An operator inequality that we make frequent use of is the “projector trick inequality”:

$$\Pi_{\rho_{x^n}, \delta} \leq 2^{n[H(Y|X) + \delta]} \rho_{x^n}.$$

This follows from (12) and the fact that $\Pi_{\rho_{x^n}, \delta} \rho_{x^n} \Pi_{\rho_{x^n}, \delta} \leq \rho_{x^n}$.

B Weak Joint Typicality and the Indicator Function Trick

Consider the joint random variable (X, Y) with distribution $p(x, y)$. We define the weak jointly typical set \mathcal{T}^n as the set of all sequences whose sample joint entropy is close to the true entropy and whose marginal sample entropies are close to the true marginal entropies:

$$\mathcal{T}^n \equiv \{ (x^n, y^n) : |\overline{H}(x^n) - H(X)| \leq \delta, |\overline{H}(y^n) - H(Y)| \leq \delta, |\overline{H}(x^n, y^n) - H(XY)| \leq \delta \}.$$

This definition implies the following inequalities:

$$\begin{aligned} \mathcal{I}((x^n, y^n) \in \mathcal{T}^n) &\leq p(x^n) 2^{n[H(X) + \delta]}, \\ \mathcal{I}((x^n, y^n) \in \mathcal{T}^n) &\leq p(y^n) 2^{n[H(Y) + \delta]}, \\ \mathcal{I}((x^n, y^n) \in \mathcal{T}^n) &\leq p(x^n, y^n) 2^{n[H(XY) + \delta]}. \end{aligned}$$

Additionally, the restrictions on the marginal and joint sample entropies imply that the following inequality holds for any $(x^n, y^n) \in \mathcal{T}^n$:

$$|\overline{H}(x^n|y^n) - H(X|Y)| \leq \delta',$$

for some δ' that is a function of δ . In turn, this inequality implies a different “indicator function trick”:

$$\mathcal{I}((x^n, y^n) \in \mathcal{T}^n) \leq p(x^n|y^n) 2^{n[H(X|Y) + \delta']}.$$

C Gentle Operator Lemma

Lemma 2 (Gentle Operator Lemma for Ensembles [28, 19, 27]) *Given an ensemble $\{p_X(x), \rho_x\}$ with expected density operator $\rho \equiv \sum_x p_X(x) \rho_x$, suppose that an operator Λ such that $I \geq \Lambda \geq 0$ succeeds with high probability on the state ρ :*

$$\text{Tr}\{\Lambda\rho\} \geq 1 - \epsilon.$$

Then the subnormalized state $\sqrt{\Lambda}\rho_x\sqrt{\Lambda}$ is close in expected trace distance to the original state ρ_x :

$$\mathbb{E}_X \left\{ \left\| \sqrt{\Lambda}\rho_X\sqrt{\Lambda} - \rho_X \right\|_1 \right\} \leq 2\sqrt{\epsilon}.$$

D Marginal and Conditional Distributions formed from the Source Distribution

In this appendix, we detail several properties of the source distribution $p(u, v, w)$, under the assumption that W is the common part of $(U, V) \sim p(u, v)$ (as described in the main text). First, we demonstrate the following equality:

$$\sum_w p(u, w|v) = p(u|v)$$

Consider that

$$\begin{aligned} \sum_w p(u, w|v) &= \sum_w \frac{p(u, w, v)}{p(v)} \\ &= \sum_w \frac{p(u, v) \delta_{w, f(u)} \delta_{w, g(v)}}{p(v)} \\ &= \frac{p(u, v) \sum_w \delta_{w, f(u)} \delta_{w, g(v)}}{p(v)} \\ &= \frac{p(u, v)}{p(v)} \\ &= p(u|v) \end{aligned} \tag{13}$$

Similarly, we have

$$\sum_w p(v, w|u) = p(v|u). \tag{14}$$

We would like to show that

$$p(v|u, w) = p(v|u),$$

which seems reasonable since knowledge of u and w should be the same as knowledge of u alone (given that w is computed from u). We have

$$\begin{aligned} p(v|u, w) &= \frac{p(u, v, w)}{p(u, w)} \\ &= \frac{p(w|u, v) p(u, v)}{p(w|u) p(u)} \\ &= \frac{\delta_{w, f(u)} \delta_{w, g(v)} p(v|u)}{\delta_{w, f(u)}} \end{aligned}$$

This distribution is only defined whenever $w = f(u)$.

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